

A remark on the Lagrange structure of the unfolded field theory

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Abstract

Any local field theory can be equivalently reformulated in the so-called unfolded form. General unfolded equations are non-Lagrangian even though the original theory is Lagrangian. Using the theory of a scalar field as a basic example, the concept of Lagrange anchor is applied to perform a consistent path-integral quantization of unfolded dynamics. It is shown that the unfolded representation for the canonical Lagrange anchor of the d'Alembert equation inevitably involves an infinite number of space-time derivatives.

1 Introduction

Any system of local field equations can be equivalently reformulated in the so-called unfolded form [1], [2]. The field content of the unfolded formulation includes, besides the original fields, an infinite collection of auxiliary fields absorbing all the space-time derivatives of the original fields. The unfolded equations are generally not Lagrangian even though the system has been Lagrangian before unfolding. One of the remarkable achievements of the unfolded formalism is that it is the only known form of the interacting higher-spin field theories, see [3], [4] for reviews. On the other hand, the absence of an action functional is usually regarded a serious disadvantage of the unfolded dynamics, especially when the quantization problem is the issue.

In [5], a new quantization method was proposed for not necessarily Lagrangian systems. The main geometric ingredient of the method is the notion of a Lagrange anchor. The existence of a compatible Lagrange anchor appears to be a less restrictive condition for the

field equations than the requirement to admit a variational formulation. Given a Lagrange anchor, the system can be quantized in three different ways: (i) by converting the original d -dimensional field theory into an equivalent topological Lagrangian model in $d + 1$ dimensions [5], (ii) by constructing a generalized Schwinger-Dyson equation, which defines the generating functional of Green's functions [6], (iii) by imbedding the original dynamics into an augmented Lagrangian model in d dimensions [7]. The Lagrange anchor also connects symmetries to conservation laws [8], extending the Noether theorem beyond the class of Lagrangian dynamics. Many explicit examples of Lagrange anchors are known for various non-Lagrangian field theories [5]-[10].

In this paper we study the structure of Lagrange anchors compatible with the unfolded form of dynamics. In particular, we address the case where the unfolded field theory admits an equivalent Lagrangian formulation with finite number of fields. In this case, the general structure of the Lagrange anchor was first established in [11]. In the present paper, to avoid technicalities and to illuminate the main properties of the unfolded Lagrange anchors, we focus on the theory of a single scalar field. The general constructions and conclusions, being derived for this quite an elementary model, remain applicable to a much broader class of dynamics.

The paper is organized as follows. The next section starts with recalling the free-differential-algebra approach to the formulation of classical field theory and its relation to the unfolded formalism. Then we briefly describe the on-shell and off-shell unfolded representations for the d'Alembert equation. Section 3 contains a brief exposition of basic definitions, constructions and motivations concerning the Lagrange anchor and its application to the path-integral quantization of (non-)Lagrangian field theories. In Section 4, we identify a natural Lagrange anchor for the unfolded representation of the d'Alembert equation. It is the anchor which provides the equivalence of the quantized unfolded model to the standard path-integral quantization of the scalar field. In Section 5, we prove a no-go theorem for the existence of Lagrange anchors without space-time derivatives in the on-shell unfolded formalism. In Section 6, we summarize the results and discuss possible applications of the proposed Lagrange anchor construction to the unfolded form of field theory.

2 Unfolded representation of the d'Alembert equation

We start with a very brief review of the free-differential-algebra (FDA) approach to the formulation of general covariant field theories [12], [13]. In the FDA formalism one deals with a multiplet $\{w^a\}$ of differential forms of various degrees on the space-time manifold M . The field equations are of the form

$$dw^a = Q^a(w). \quad (2.1)$$

The r.h.s. of these equations, Q 's, are given by exterior polynomials in w 's. The identity $d^2 = 0$ for the exterior differential results in the following compatibility conditions for equations (2.1):

$$Q^a \frac{\partial Q^b}{\partial w^a} = 0. \quad (2.2)$$

These conditions are assumed to be identically satisfied for all w 's. Equation (2.2) has the following geometric interpretation. The forms w^a can be treated as coordinates on an \mathbb{N} -graded manifold \mathcal{M} with the degrees of coordinates being the corresponding form-degrees. Then the r.h.s of (2.1) defines a *homological vector field* $Q = Q^a \partial_a$ on \mathcal{M} , i.e., a smooth vector field of degree 1 squaring to zero (2.2). Besides an explicit invariance under the diffeomorphisms of M , equations (2.1) enjoy the gauge symmetries

$$\delta_\varepsilon w^a = d\varepsilon^a - \frac{\partial Q^a}{\partial w^b} \varepsilon^b, \quad (2.3)$$

where the gauge parameters ε^a are the forms of appropriate degrees.

In the field theory, it can be useful to consider non-free differential algebras, when the differential equations (2.1) are supplemented by a set of algebraic constraints

$$T^A(w) = 0. \quad (2.4)$$

Being exterior polynomials in w 's, the constraints generate an ideal $J \subset \Lambda(M)$ in the algebra of all differential forms on M . To avoid further compatibility conditions, J is assumed to be a differential ideal, i.e., $dJ \subset J$. On account of (2.1), the last requirement amounts to

$$Q^a \frac{\partial T^A}{\partial w^a} = U_B^A T^B \quad (2.5)$$

for some exterior polynomials $U_B^A(w)$. Under certain regularity conditions, the algebraic constraints (2.4) define a smooth submanifold $\mathcal{N} \subset \mathcal{M}$ and condition (2.5) says that the homological vector field Q is tangent to \mathcal{N} . Restricting the vector field Q to \mathcal{N} , e.g. by explicitly solving the algebraic constraints, we arrive to another FDA with fewer generators.

By unfolding (or unfolded representation of) a local field theory, one usually means a reformulation of the original field equations in the form (2.1), (2.4). Such a reformulation usually involves an infinite number of auxiliary fields. In what follows we will distinguish between *off-shell* and *on-shell* unfolded representations. In the former case, there are both the differential equations (2.1) and the algebraic constraints (2.4) (also called shell conditions). In the on-shell formulation, the fields w are unconstrained, that can be a result of solving the algebraic constraints of the off-shell formulation.

In this paper we will mostly consider the unfolded representation for the free, massless, scalar field theory in d -dimensional Minkowski space, though our constructions and conclusions remain applicable to a much broader class of field theories. The standard formulation

of the scalar field theory is based on the d'Alembert equation

$$\square\varphi = 0. \quad (2.6)$$

Loosely, the idea behind constructing the unfolded representation (2.1), (2.4) for a given system of differential equations is to introduce an infinite collection of auxiliary fields absorbing all the derivatives of the original fields. This procedure is known as the infinite jet prolongation [14]. In terms of the extended set of fields, the system of differential equations (2.1) appears as the definition of the jet prolongation (this is called a contact system), whereas the original field equations along with all their differential consequences turn into the algebraic constraints (2.4) in the jet space. To keep the system explicitly invariant under diffeomorphisms, the unfolded representation is formulated in a general frame, even though the space-time M can be flat (as is supposed throughout this paper). As usual, this implies introduction of a vielbein $e^a \in \Lambda^1(M)$ and a compatible Lorentz connection $\omega^{ab} = -\omega^{ba} \in \Lambda^1(M)$. All the Lorentz indices a, b, c, \dots are raised and lowered with Minkowski metric η_{ab} .

The unfolded version of the d'Alembert equation (2.6) reads [4], [15]:

$$De^a = 0, \quad d\omega^a_b = -\omega^a_c \wedge \omega^c_b, \quad (2.7)$$

$$D\varphi_{a_1 \dots a_s} = e^a \varphi_{aa_1 \dots a_s}, \quad s = 0, 1, \dots, \quad (2.8)$$

$$\varphi^a_{aa_1 \dots a_s} = 0, \quad s = 0, 1, \dots. \quad (2.9)$$

Here $D = e^a \wedge \nabla_a = d + \omega$ is the Lorenz-covariant differential and $\{\varphi_{a_1 \dots a_s}\}$ is an infinite collection of fully symmetric Lorentz tensors.

Let us comment on the structure of the unfolded equations. The first equation in (2.7) is the usual zero-torsion condition. It allows one to express the components of the Lorentz connection ω^{ab} via the vielbein field e^a . The second equation in (2.7) is the zero-curvature condition for the Lorentz connection. So, there is a coordinate system $\{x^a\}$ on M in which $\omega^{ab} = 0$ and $e^a = dx^a$. This also means that the 1-form fields ω^{ab} and e^a are pure gauge. Equations (2.8) can be viewed as a reparametrization invariant form of the *contact system* [14]. By taking linear combinations these equations can be rearranged into the form

$$\varphi_{a_1 \dots a_s} = \nabla_{a_1} \dots \nabla_{a_s} \varphi. \quad (2.10)$$

So, the contact equations just define the auxiliary fields $\varphi_{a_1 \dots a_s}$, $s > 0$, in terms of successive derivatives (infinite jet) of a single scalar field φ .

Substituting (2.10) into (2.9), we get the sequence of equations

$$\nabla_{a_1} \dots \nabla_{a_s} \square\varphi = 0, \quad \square = \nabla^a \nabla_a, \quad s = 0, 1, 2, \dots, \quad (2.11)$$

which is clearly equivalent to the original d'Alembert's equation (2.6). The unfolded system is obviously consistent, because applying d to both sides of (2.7), (2.8) and (2.9) does not lead to any new condition. The differential algebra underlying the unfolded formulation above is generated by the finite set of 1-forms $\{e^a, \omega^{ab}\}$ and the infinite collection of 0-forms $\{\varphi_{a_1 \dots a_s}\}$. In view of the algebraic constraints (2.9), it is not a free algebra. To pass from the off-shell to on-shell formulation one only has to assume the Lorentz tensors $\varphi_{a_1 \dots a_s}$ to be traceless.

For the field equations (2.7)-(2.9) the general gauge transformations (2.3) read

$$\begin{aligned} \delta_\varepsilon \varphi_{a_1 \dots a_s} &= \varepsilon^a \varphi_{aa_1 \dots a_s} - \sum_{i=1}^s \varepsilon_{a_i}^a \varphi_{a_1 \dots a_{i-1} a a_{i+1} \dots a_s}, \\ \delta_\varepsilon e^a &= D\varepsilon^a - \varepsilon^{ab} e_b, \quad \delta_\varepsilon \omega^{ab} = D\varepsilon^{ab}. \end{aligned} \quad (2.12)$$

The gauge parameters ε^a and $\varepsilon^{ab} = -\varepsilon^{ba}$ correspond, respectively, to the general coordinate transformations and the local Lorentz rotations.

Notice that the subsystem (2.7), (2.8), being considered separately from constraints (2.9), remains formally consistent. This truncated system, however, is dynamically empty: It just expresses the higher rank Lorentz tensors $\varphi_{a_1 \dots a_s}$ in terms of the unconstrained scalar field φ , and its derivatives, according to (2.10). As a result, the truncated system of equations possesses the extra gauge symmetry of the form

$$\delta_\varepsilon \varphi_{a_1 \dots a_s} = \nabla_{a_1} \dots \nabla_{a_s} \varepsilon. \quad (2.13)$$

Combining these transformations with (2.12), one can gauge out all the fields. Also notice that the transformations (2.13) involve s derivatives of the gauge parameter for the field $\varphi_{a_1 \dots a_s}$. From this viewpoint, these transformations are local, as the number of derivatives is finite for every single field. However, the order of the derivatives is unbounded, because it is growing with the order of the jet, whereas the jet prolongation is infinite in the unfolded formalism. We will return to this point in Section 4. The algebraic condition (2.9) breaks the gauge symmetry (2.13), so the unfolded system gets back its physical degrees of freedom and becomes equivalent to the d'Alembert equation as soon as the Lorentz tensors $\varphi_{a_1 \dots a_s}$ are constrained to be traceless.

It is worth noting that the unfolded system (2.7)-(2.9) is not Lagrangian in any dimension unlike the d'Alembert equation (2.6). Furthermore, it remains non-Lagrangian even if one omits equations (2.7), (2.9) by imposing gauge fixing conditions on the vielbein and the Lorentz connection and assuming the auxiliary fields $\varphi_{a_1 \dots a_s}$ to be traceless. This fact may seem restricting the use of the unfolded formalism especially when it comes to quantizing the classical fields. Actually, the Lagrangian formalism is a special case of a more general concept called a Lagrange structure [5]. The existence of a compatible Lagrange structure is much less restrictive for the field equations than the requirement to admit a Lagrangian.

Given a Lagrange structure, it is still possible to consistently quantize the classical theory [5], [6], [7], even though it admits no Lagrangian. The Lagrange structure also connects symmetries with conservation laws [8]. The next section contains a simplified exposition of the Lagrange structure and the corresponding quantization method. This description is sufficient for many field-theoretical applications, including the issues of this paper. A more systematic and rigorous presentation can be found in [5]-[8].

3 A generalized Schwinger-Dyson equation

In the covariant formulation of quantum field theory one usually studies the path integrals of the form

$$\langle \mathcal{O} \rangle = \int [d\varphi] \mathcal{O}[\varphi] e^{\frac{i}{\hbar} S[\varphi]}. \quad (3.1)$$

After normalization, the integral defines the quantum average of an observable $\mathcal{O}[\varphi]$ in the theory with action $S[\varphi]$. It is believed that evaluating the path integrals for various observables \mathcal{O} , one can extract all physically relevant information about the quantum dynamics of the model.

The functional $\Psi[\varphi] = e^{\frac{i}{\hbar} S[\varphi]}$, weighting the contribution of a particular field configuration φ to the quantum average, is known as the Feynman probability amplitude on the configuration space of fields. This amplitude can be defined as a unique (up to a normalization factor) solution to the Schwinger-Dyson (SD) equation¹

$$\left(\frac{\partial S}{\partial \varphi^i} + i\hbar \frac{\partial}{\partial \varphi^i} \right) \Psi[\varphi] = 0. \quad (3.2)$$

Performing the Fourier transform from the fields φ to their sources $\bar{\varphi}$, we can bring (3.2) to a more familiar form

$$\left(\frac{\partial S}{\partial \varphi^i}(\bar{\varphi}) - \bar{\varphi}_i \right) Z[\bar{\varphi}] = 0, \quad \bar{\varphi}^i \equiv i\hbar \frac{\partial}{\partial \bar{\varphi}_i}, \quad (3.3)$$

where

$$Z[\bar{\varphi}] = \int [d\varphi] e^{\frac{i}{\hbar} (S - \bar{\varphi} \varphi)} \quad (3.4)$$

is the generating functional of Green's functions.

The following observations provide guidelines for the generalization of the Schwinger-Dyson equation to non-Lagrangian field theory.

(i) Although the Feynman probability amplitude involves an action functional, the SD equations (3.2) contain solely the classical field equations, not the action as such.

¹Here we use the condensed notation, so that the partial derivatives with respect to fields should be understood as variational ones.

(ii) In the classical limit $\hbar \rightarrow 0$, the second term in the SD equation (3.2) vanishes, and the Feynman probability amplitude Ψ turns into the Dirac distribution supported at the classical solutions to the field equations. Formally, $\Psi[\varphi]|_{\hbar \rightarrow 0} \sim \delta[\partial_i S]$ and one can think of the last expression as the classical probability amplitude.

(iii) It is quite natural to treat the sources $\bar{\varphi}$ as the momenta canonically conjugate to the fields φ , so that the only non-vanishing Poisson brackets are $\{\varphi^i, \bar{\varphi}_j\} = \delta_j^i$. Then, one can regard the SD operators

$$\frac{\partial S}{\partial \varphi^i} + i\hbar \frac{\partial}{\partial \varphi^i} \quad (3.5)$$

involved in (3.2) as resulting from the canonical quantization of the first class constraints

$$\partial_i S[\varphi] - \bar{\varphi}_i \approx 0 \quad (3.6)$$

on the phase space of fields and sources. Upon this interpretation, the Feynman probability amplitude describes a unique physical state of a first-class constrained theory. This state is unique because the “number” of the first class constraints (3.6) is equal to the “dimension” of the configuration space of fields. Quantizing the constrained system (3.6) in the momentum representation yields the SD equation (3.3) for the partition function $Z[\bar{\varphi}]$.

The above interpretation of the SD equations as operator first class constraints on a physical wave-function suggests a direct way to their generalization. Consider a set of field equations

$$T_a(\varphi^i) = 0, \quad (3.7)$$

which do not necessarily follow from the variational principle. In this case, the (discrete parts of) superindices a and i may run over different sets. Proceeding from the heuristic arguments above, we can take the following ansatz for the $\varphi\bar{\varphi}$ -symbols of the Schwinger-Dyson operators:

$$\mathcal{T}_a = T_a(\varphi) - V_a^i(\varphi)\bar{\varphi}_i + O(\bar{\varphi}^2). \quad (3.8)$$

The symbols are defined as formal power series in sources $\bar{\varphi}$ with leading terms being the classical equations of motion. Requiring the Hamiltonian constraints $\mathcal{T}_a \approx 0$ to be first class, i.e.,

$$\{\mathcal{T}_a, \mathcal{T}_b\} = U_{ab}^c \mathcal{T}_c, \quad U_{ab}^c(\varphi, \bar{\varphi}) = C_{ab}^c(\varphi) + O(\bar{\varphi}), \quad (3.9)$$

we obtain an infinite set of relations on the expansion coefficients of \mathcal{T}_a in the powers of sources. In particular, verifying the involution relations (3.9) up to zero order in $\bar{\varphi}$, we find

$$V_a^i \partial_i T_b - V_b^i \partial_i T_a = C_{ab}^c T_c. \quad (3.10)$$

The value $V_a^i(\varphi)$ defined by (3.10) is called the *Lagrange anchor*. The entire sequence of the expansion coefficients defines the *Lagrange structure*.

For variational field equations, $T_a = \partial_i S$, one can set the Lagrange anchor to be the unit matrix $V_a^i = \delta_a^i$. This choice results in the standard Schwinger-Dyson operators (3.5) obeying the abelian involution relations. Generally, the Lagrange anchor may be field-dependent and/or noninvertible. If the Lagrange anchor is invertible (in which case the number of equations must coincide with the number of fields), then the operator V^{-1} plays the role of integrating multiplier in the inverse problem of calculus of variations. So, the existence of the invertible Lagrange anchor amounts to the existence of action. The other extreme choice, $V = 0$, is always possible and corresponds to the classical probability amplitude $\Psi[\varphi] \sim \delta[T_a(\varphi)]$ supported at the classical solutions.

In the non-Lagrangian case, the constraints (3.8) are not generally the whole story. The point is that the number of (independent) field equations can happen to be less than the dimension of the configuration space of fields. In that case, the field equations (3.7) do not specify a unique solution with prescribed boundary conditions or, stated differently, the system enjoys a gauge symmetry generated by an on-shell integrable vector distribution $R_\alpha = R_\alpha^i(\varphi)\partial_i$ such that

$$R_\alpha^i \partial_i T_a = U_{\alpha a}^b T_b, \quad [R_\alpha, R_\beta] = U_{\alpha\beta}^\gamma R_\gamma + T_a U_{\alpha\beta}^{ai} \partial_i \quad (3.11)$$

for some structure functions $U_{\alpha a}^b(\varphi)$ and $U_{\alpha\beta}^{ai}(\varphi)$. To take the gauge invariance into account at quantum level, one has to impose additional first class constraints on the fields and sources. Namely,

$$\mathcal{R}_\alpha = R_\alpha^i(\varphi)\bar{\varphi}_i + O(\bar{\varphi}^2) \approx 0. \quad (3.12)$$

The leading terms of these constraints coincide with the $\varphi\bar{\varphi}$ -symbols of the gauge symmetry generators and the higher orders in $\bar{\varphi}$ are determined from the requirement the Hamiltonian constraints $\mathbb{T}_I = (\mathcal{T}_a, \mathcal{R}_\alpha)$ to be in involution². With all the gauge symmetries included, the constraint surface $\mathbb{T}_I \approx 0$ is proved to be a Lagrangian submanifold in the phase space of fields and sources and the gauge invariant probability amplitude is defined as a unique solution to the *generalized SD equation*

$$\widehat{\mathbb{T}}_I \Psi = 0. \quad (3.13)$$

The last formula is just the definition of a physical state in the Dirac quantization method of constrained dynamics. A more systematic treatment of the generalized SD equation within the BRST formalism can be found in [5], [6], [7].

In what follows we will refer to the first class constraints $\mathbb{T}_I \approx 0$ as the *Schwinger-Dyson extension* of the original equations of motion (3.7). Notice that the defining relations (3.10) for the Lagrange anchor together with the “boundary conditions” (3.8) and (3.12) do

²For a Lagrangian gauge theory we have $\mathcal{T}_i = \partial_i S - \bar{\varphi}_i$ and $\mathcal{R}_\alpha = -R_\alpha^i \mathcal{T}_i = R_\alpha^i \bar{\varphi}_i$. In this case, one may omit the “gauge” constraints $\mathcal{R}_\alpha \approx 0$ as they are given by linear combinations of the “dynamical” constraints $\mathcal{T}_i \approx 0$.

not specify a unique SD extension for a given system of field equations. One part of the ambiguity is related to canonical transformations in the phase space of fields and sources. If the generator G of a canonical transform is at least quadratic in sources,

$$G = \frac{1}{2} G^{ij}(\varphi) \bar{\varphi}_i \bar{\varphi}_j + O(\bar{\varphi}^3), \quad (3.14)$$

then the transformed constraints

$$\begin{aligned} \mathcal{T}'_a &= e^{\{G, \cdot\}} \mathcal{T}_a = T_a + (V_a^i + G^{ij} \partial_j T_a) \bar{\varphi}_i + O(\bar{\varphi}^2), \\ \mathcal{R}'_\alpha &= e^{\{G, \cdot\}} \mathcal{R}_\alpha = R_\alpha^i \bar{\varphi}_i + O(\bar{\varphi}^2) \end{aligned} \quad (3.15)$$

are in involution and start with the same equations of motion and gauge symmetry generators. Another ambiguity stems from changing the basis of the constraints:

$$\begin{aligned} \mathcal{T}''_a &= U_a^b \mathcal{T}_b + U_a^\alpha \mathcal{R}_\alpha = T_a + (V_a^i + A_a^{bi} T_b + B_a^\alpha R_\alpha^i) \bar{\varphi}_i + O(\bar{\varphi}^2), \\ \mathcal{R}''_\alpha &= U_\alpha^\beta \mathcal{R}_\beta + U_\alpha^a \mathcal{T}_a = R_\alpha^i \bar{\varphi}_i + O(\bar{\varphi}^2), \end{aligned} \quad (3.16)$$

where

$$\begin{aligned} U_a^b &= \delta_a^b + A_a^{bi} \bar{\varphi}_i + O(\bar{\varphi}^2), & U_a^\alpha &= B_a^\alpha + O(\bar{\varphi}), \\ U_\alpha^\beta &= \delta_\alpha^\beta + O(\bar{\varphi}), & U_\alpha^a &= O(\bar{\varphi}). \end{aligned} \quad (3.17)$$

Combining (3.15) with (3.16) we see that the Lagrange anchor is defined modulo the equivalence relations

$$V_a^i \sim V_a^i + T_b A_a^{bi} + B_a^\alpha R_\alpha^i + G^{ij} \partial_j T_a. \quad (3.18)$$

A more rigorous treatment of the Lagrange structure and generalized Schwinger-Dyson equation is provided by the corresponding BRST formalism [5], [6]. From the viewpoint of the BRST theory, all transformations (3.15) and (3.16) are induced by canonical transforms in the ghost-extended phase space, and the equivalence classes of Lagrange anchors are identified with certain classes of the BRST cohomology.

4 The Schwinger-Dyson extension of unfolded dynamics

In this section, the general quantization procedure described above is applied to the scalar field theory in the unfolded formulation.

Imposing the requirements of (i) space-time locality, (ii) Poincaré covariance, and (iii) linearity in fields and sources, one can see that the most general SD extension of the original d'Alembert's equation (2.6) reads

$$\square \varphi + \sum_{n=0}^N a_n \square^n \bar{\varphi} \approx 0, \quad (4.1)$$

where a_n are constants. These constraints are clearly in abelian involution. The local canonical transform

$$\varphi \rightarrow \varphi - \sum_{n=1}^N a_n \square^{n-1} \bar{\varphi}, \quad \bar{\varphi} \rightarrow \bar{\varphi}, \quad (4.2)$$

brings the first class constraints (4.1) to the form

$$\square \varphi + a_0 \bar{\varphi} \approx 0. \quad (4.3)$$

This means that the most general Lagrange anchor (4.1) is equivalent (up to normalization) to the canonical one (3.6), whenever $a_0 \neq 0$. If $a_0 = 0$ the anchor is trivial.

Unlike the d'Alembert equation, the unfolded system (2.7)-(2.9) is non-Lagrangian. Therefore, one cannot use the conventional quantization prescriptions. The unfolded equations, however, admit a quite natural SD extension, which can be viewed as an “unfoldization” of the constraints (4.3). To describe this extension, following the general procedure of Section 3, we introduce the sources \bar{e}_a and $\bar{\omega}_{ab} = -\bar{\omega}_{ba}$ for the vielbein and connection fields to be $(d-1)$ -forms on M . The fully symmetric Lorentz tensors $\bar{\varphi}^{a_1 \dots a_s}$ with values in d -forms are introduced as the sources for $\varphi_{a_1 \dots a_s}$. The Poisson brackets in the phase space of fields and sources are defined by the following symplectic form:

$$\Omega = \int_M \left(\delta e^a \wedge \delta \bar{e}_a + \delta \omega^{ab} \wedge \delta \bar{\omega}_{ab} + \sum_{s=0}^{\infty} \delta \varphi_{a_1 \dots a_s} \wedge \delta \bar{\varphi}^{a_1 \dots a_s} \right). \quad (4.4)$$

Here the sign \wedge stands for both the exterior product of differential forms on M and the exterior product of variational differentials.

Let us first consider the off-shell formulation. In this case, the differential equations (2.7) and (2.8) remain intact, while the algebraic constraints (2.9) are replaced by the expressions

$$\varphi^a_{aa_1 \dots a_s} + \nabla_{a_1} \dots \nabla_{a_s} R \approx 0, \quad (4.5)$$

where

$$R = * \sum_{n=0}^{\infty} (-1)^n \nabla_{a_1} \dots \nabla_{a_n} \bar{\varphi}^{a_1 \dots a_n}, \quad (4.6)$$

and the Hodge operator $* : \Lambda^p(M) \rightarrow \Lambda^{d-p}(M)$ is defined with respect to the space-time metric. Since the unfolded system (2.7)-(2.9) is gauge invariant we should also add the constraints associated to the gauge symmetry generators (2.12). These are given by

$$\begin{aligned} D\bar{e}_a - \sum_{s=0}^{\infty} \varphi_{aa_1 \dots a_s} \bar{\varphi}^{a_1 \dots a_s} &\approx 0, \\ D\bar{\omega}^{ab} - e^{[a} \wedge \bar{e}^{b]} + \sum_{s=0}^{\infty} (s+1) \bar{\varphi}^{a_1 \dots a_s [a} \varphi^{b]}_{a_1 \dots a_s} &\approx 0. \end{aligned} \quad (4.7)$$

It is clear that the Hamiltonian constraints (2.7), (2.8), and (4.7) are of the first class. Then, it remains to prove the involution of constraints (4.5) among themselves and with the other constraints. Observe that the SD constraints (4.5) are explicitly covariant under the action of diffeomorphisms and the local Lorentz transformations generated by (2.12). So, the Poisson brackets of (4.7) and (4.5) must be proportional to (4.5). To prove the involution of (2.7), (2.8) and (4.5), we notice that the Hamiltonian flow $\{R, \cdot\}$ generates the gauge transformations (2.13) for the truncated system (2.7), (2.8). Hence, the constraints (4.5) Poisson commute with (2.7) and (2.8). Finally, we know that the first class constraints (2.8) are equivalent to (2.10). Taking into account (2.10), we can replace (4.5) with the equivalent set of constraints

$$\nabla_{a_1} \cdots \nabla_{a_s} \left((\square\varphi + *\bar{\varphi}) + * \sum_{n=1}^{\infty} (-1)^n \nabla_{a_1} \cdots \nabla_{a_n} \bar{\varphi}^{a_1 \cdots a_n} \right) \approx 0. \quad (4.8)$$

The abelian involution of the constraints (4.8), and hence (4.5), is now obvious.

Let us summarize the interim result. The system of equations (2.7), (2.8), (4.5), (4.7) defines the complete set of the Schwinger-Dyson constraints for the off-shell unfolded formulation of scalar field's dynamics. Below, we will see that the corresponding probability amplitude leads to the quantum theory, which is completely equivalent to the conventional theory based on the d'Alembert equation.

Two steps are needed to put the unfolded system (2.7), (2.8), (4.5), (4.7) on-shell. First, we express the traces of the tensor fields $\varphi_{a_1 \cdots a_s}$ via the sources according to (4.5) and substitute them into the constraints (2.7), (2.8), (4.7). Constraints (4.5) are now identically satisfied, and the remaining SD constraints are still in involution. As the second step, we set the trace part of the sources $\bar{\varphi}^{a_1 \cdots a_s}$ to zero. Again, this cannot break involution as the constraints derived at the first step do not involve the traces of the fields $\varphi_{a_1 \cdots a_s}$. Denote by $\phi_{a_1 \cdots a_s}$ and $\bar{\phi}^{a_1 \cdots a_s}$ the traceless parts of the tensors $\varphi_{a_1 \cdots a_s}$ and $\bar{\varphi}^{a_1 \cdots a_s}$. Then we arrive at the following SD constraints:

$$De^a \approx 0, \quad d\omega^a_b + \omega^a_c \wedge \omega^c_b \approx 0, \quad D\Phi_{a_1 \cdots a_s} - e^a \Phi_{aa_1 \cdots a_s} \approx 0, \quad (4.9)$$

$$D\bar{e}_a - \sum_{s=0}^{\infty} \Phi_{aa_1 \cdots a_s} \bar{\phi}^{a_1 \cdots a_s} \approx 0, \quad (4.10)$$

$$D\bar{\omega}^{ab} - e^{[a} \wedge \bar{e}^{b]} + \sum_{s=0}^{\infty} (s+1) \bar{\phi}^{a_1 \cdots a_s [a} \phi^{b]}_{a_1 \cdots a_s} \approx 0, \quad (4.11)$$

where

$$\Phi_{a_1 \cdots a_s} = \phi_{a_1 \cdots a_s} - \sum_{k=1}^{[s/2]} \alpha_k^s \eta_{(a_1 a_2} \cdots \eta_{a_{2k-1} a_{2k}} \nabla_{a_{2k+1}} \cdots \nabla_{a_s} \square^{k-1} \tilde{R}, \quad (4.12)$$

$$\tilde{R} = * \sum_{k=0}^{\infty} (-1)^k \nabla_{a_1} \cdots \nabla_{a_k} \bar{\phi}^{a_1 \cdots a_k}, \quad (4.13)$$

and the coefficients α_k^s are determined by the recurrent relations

$$\alpha_1^s = \frac{1}{d+2s-4}, \quad \alpha_k^s = \frac{\alpha_{k-1}^s}{2k+2-d-2s}. \quad (4.14)$$

The round brackets in (4.12) mean summation over all inequivalent permutations of the indices enclosed. Notice that the SD constraints (4.10) involve bilinear combinations of sources, cf. (3.12). On account of (4.9) we can replace (4.10) with the equivalent constraints

$$D\bar{e}_a - \sum_{s=0}^{\infty} \bar{\phi}^{a_1 \cdots a_s} \nabla_a \phi_{a_1 \cdots a_s} \approx 0. \quad (4.15)$$

Equations (4.9), (4.11), (4.15) define an equivalent basis of the SD constraints that are at most linear in sources. These relations define the Lagrange structure for the on-shell unfolded formulation of the scalar field.

From the viewpoint of the constrained Hamiltonian dynamics, the transition from the off-shell to the on-shell formulation is the Hamiltonian reduction by the second class constraints

$$\bar{\varphi}^a_{aa_1 \cdots a_s} \approx 0, \quad \varphi^a_{aa_1 \cdots a_s} + \nabla_{a_1} \cdots \nabla_{a_s} R \approx 0, \quad (4.16)$$

and the canonical Poisson brackets of the traceless fields and sources $\{\phi_{a_1 \cdots a_s}, \bar{\phi}^{b_1 \cdots b_k}\}$ appear as Dirac's brackets in the reduced phase space.

Let us comment on the unusual property of the SD constraints derived above both in the off-shell and on-shell formulations. The unfolded field equations involve infinitely many fields, as the jet prolongation is infinite in this formalism. The SD extensions (4.9), (4.11), (4.15) of the field equations involve a finite number of derivatives for every single field and source, though the order of derivatives increases with the order of jet. This fact directly follows from the structure of the Hamiltonian generator (4.6) for the gauge transformations (2.13) of the contact system. As we will see in the next section this property is unavoidable for the Lagrange anchor in the unfolded formalism.

Let us check that the quantization of the unfolded system results in the standard Feynman's probability amplitude for the free massless scalar field. Given the off-shell constraints (2.7), (2.8), (4.5), (4.7), the generalized SD equations for the quantum probability amplitude (3.13) are readily constructed following the general prescriptions of Section 3. They are satisfied by the functional

$$\Psi \sim e^{\frac{i}{\hbar} S[\varphi]} \delta(de + \omega \wedge e) \delta(d\omega + \omega \wedge \omega) \prod_{s=0}^{\infty} \delta(D\varphi_{a_1 \cdots a_s} - e^a \varphi_{aa_1 \cdots a_s}), \quad (4.17)$$

where

$$S[\varphi] = \frac{1}{2} \int_M d\varphi \wedge * d\varphi \quad (4.18)$$

is the usual action for the free massless scalar field in general gravitational background.

As is seen, no quantum fluctuations arise for the holonomic constraints, i.e., the equations without source extensions. With the δ -functions of the contact equations all the auxiliary fields $\varphi_{a_1 \dots a_s}$ can be integrated out in the path integral. The vielbein and the Lorentz connection also remain classical fields with no quantum fluctuations. As they are pure gauge, their contribution is eliminated from the path integral by imposing gauge-fixing conditions. This yields the standard Feynman's amplitude for the original scalar field. The fact that the Schwinger-Dyson equations define a “smeared” probability amplitude (rather than a δ -distribution) confirms that the constructed Lagrange anchor is nontrivial.

5 Algebraic Lagrange anchors

The on-shell unfolded representation for the dynamics of scalar field (2.7), (2.8) suggests to consider a general FDA generated by finite or countable sets of 1-forms $\{\theta^a\}$ and 0-forms $\{\phi^i\}$. With this field content, the general FDA equations (2.1) take the form

$$d\phi^i = A_a^i(\phi)\theta^a, \quad d\theta^a = C_{bc}^a(\phi)\theta^b \wedge \theta^c, \quad (5.1)$$

where A 's and C 's are smooth functions of scalar fields ϕ^i . In view of (2.2) these structure functions obey the relations

$$[A_a, A_b] = C_{ab}^c A_c, \quad C_{[ab}^d C_{c]d}^m + \partial_i C_{[ab}^m A_{c]}^i = 0, \quad (5.2)$$

where the values $A_a = A_a^i(\phi) \frac{\partial}{\partial \phi^i}$ are viewed as vector fields on the ϕ -space and the square brackets stand for antisymmetrization of the indices enclosed. For linearly independent vector fields A_a the second equation in (5.2) follows from the first one by the Jacobi identity for the Lie bracket of vector fields. From the viewpoint of the target-space geometry, relations (5.2) define a Lie algebroid with anchor A .

The local BRST cohomology in the theories of type (2.1) has been recently studied in [10]. It was shown that under certain assumptions this cohomology is isomorphic to the Q -cohomology on the target space \mathcal{M} . In particular, every Lagrange anchor was proven to be equivalent to the anchor without space-time derivatives of fields and sources (the algebraic Lagrange anchor). The proof assumed implicitly that the order of the derivatives is bounded from above for all fields and sources. This assumption does not follow from the fact that the order of derivatives is finite for every single field or source, whenever the number of fields is infinite. Hence, only two possibilities are admissible: (i) the Lagrange anchor (4.9) is equivalent to some algebraic anchor or (ii) the equivalence class of the Lagrange anchor (4.9) does not have any representative with bounded order of derivatives. Below, we prove that the on-shell unfolded representation does not admit an algebraic Lagrange anchor

whenever $d > 3$. This means that the unbounded order of derivatives in (4.9)-(4.11) has no alternative.

Proposition 5.1. *The system of differential equations (5.1) admits no algebraic Lagrange anchor (except zero) whenever $d > 3$.*

Proof. Let x^μ be local coordinates on M . Denote by $\bar{\phi}_i$ and $\bar{\theta}_a$ the conjugate sources. It is convenient to think of $\bar{\phi}_i$ and $\bar{\theta}_a$ as the sets of scalar and vector fields on M . The canonical symplectic form on the phase space of fields and sources reads

$$\Omega = \int_M v(\delta\phi^i \wedge \delta\bar{\phi}_i + \delta\theta_\mu^a \wedge \delta\bar{\theta}_a^\mu), \quad (5.3)$$

where v is a suitable volume form on M and \wedge stands for the exterior product of variational differentials. The most general SD extension of equations (5.1) by means of an algebraic Lagrange anchor looks like

$$\begin{aligned} d\phi^i - A_a^i \theta^a + (U_{\mu\nu}^{ia} \bar{\theta}_a^\nu + Y_\mu^{ij} \bar{\phi}_j) dx^\mu + \dots &= 0, \\ d\theta^a - C_{bc}^a \theta^b \wedge \theta^c + (V_{\mu\nu\lambda}^{ab} \bar{\theta}_b^\lambda + W_{\mu\nu}^{ai} \bar{\phi}_i) dx^\mu \wedge dx^\nu + \dots &= 0, \end{aligned} \quad (5.4)$$

where the structure functions U, Y, V, W depend on ϕ^i and θ_μ^a , and the dots stand for possible terms of higher degrees in sources. Substituting the ansatz (5.4) to the defining relations (3.10) for the Lagrange anchor, we find that a necessary condition for these relations to be satisfied is that the following differential operators have to vanish:

$$Y_\mu^{ij} \partial_\nu + Y_\nu^{ji} \partial_\mu = 0, \quad (5.5)$$

$$W_{\mu\nu}^{ai} \partial_\tau + U_{\tau[\mu}^{ia} \partial_{\nu]} = 0, \quad (5.6)$$

$$V_{\mu\nu[\tau}^{ab} \partial_{\sigma]} + V_{\tau\sigma[\mu}^{ba} \partial_{\nu]} = 0. \quad (5.7)$$

For $d > 3$ this is only possible if all the structure functions are equal to zero. Indeed, setting $\mu \neq \nu$ in (5.5), we conclude that the first and second summands vanish separately, therefore $Y_\mu^{ij} = 0$. For $\tau \neq \mu$ and $\tau \neq \nu$ equation (5.6) yields $W_{\mu\nu}^{ai} = 0$ for all μ, ν , and hence $U_{\tau\mu}^{ia} = 0$. Finally, we let the indices μ, ν, τ, σ to be pair-wise different. Then it follows from (5.7) that the corresponding structure functions $V_{\mu\nu\tau}^{ab}$ are equal to zero. If now two space-time indices coincide (e.g. $\mu = \tau$), then we can always choose $\nu \neq \mu$ and $\nu \neq \sigma$, which yields $V_{\mu\sigma\mu}^{ab} = 0$ for all μ, σ . Thus, for all values of indices $V_{\mu\nu\tau}^{ab} = 0$. As is seen the crucial point of our argumentation is the possibility to choose four different values for the space-time indices, which is only possible if $d > 3$. \square

As an immediate consequence of Proposition 5.1 we have that the *on-shell* unfolded formulation for the scalar field theory (2.7), (2.8) admits no algebraic Lagrange anchor. The analysis above can be easily extended to the FDAs generated by forms of arbitrary degrees.

Proposition 5.2. *Let p be the highest degree of the forms entering equations (2.1). Then these equations admit no algebraic Lagrange anchor whenever $d \geq 2p + 2$.*

Notice that the low bound for $\dim M$ established by Proposition 5.2 is sharp. For if $p < d < 2p + 2$, we have the sequence of Lagrangian models

$$S = \int_M B \wedge dH$$

for the p -form H and the $(d - p - 1)$ -form B . The corresponding equations of motion,

$$dH = 0, \quad dB = 0,$$

being Lagrangian, admit the canonical Lagrange anchor (3.6), which is obviously algebraic.

6 Concluding remarks

The unfolded formalism has many remarkable properties, though the general unfolded field equations are not Lagrangian. In the standard unfolded form, even the scalar field equations do not have any Lagrangian. The Lagrange anchor can exist, however, for non-Lagrangian dynamics. Any nontrivial Lagrange anchor gives a quantization of the dynamics and can connect symmetries to conserved currents. So, the existence of the Lagrange anchor can compensate for the most important disadvantages of the unfolded formalism related to the non-variational structure of the field equations. In this paper, we studied the structure of Lagrange anchors admitted by the unfolded formalism.

As we have seen, an important feature of the Lagrange anchor in the unfolded formalism is that it is a differential operator, in general. For every single field, the order of the operator is finite. In the unfolded formalism, however, the set of fields is infinite. For example, in the scalar field theory, the set of fields includes the original scalar field together with its infinite jet prolongation. The order of derivatives is growing in the Lagrange anchor with the order of corresponding jet. In this sense the order of derivatives is unbounded. We also see that the equivalence class of the Lagrange anchor does not include any representative without derivatives.

The explicit construction of the Lagrange anchor for the unfolded dynamics of scalar field can be basically repeated for any other model admitting an equivalent formulation in terms of the finite set of fields with any Lagrange anchor, canonical or not. Some interesting field equations are formulated in the unfolded form from the outset, not being equivalent to any local field theory with a finite set of fields. An important example of this type is provided by the interacting higher-spin fields [2], [3], [4]. Our construction can be applied,

in principle, for finding the Lagrange anchor even in this case. The free limit of the higher-spin equations is known to be equivalent to a Lagrangian field theory with a finite set of fields for every spin. Hence, our procedure is applicable for the unfolded equations of higher spin free fields, and it should lead to a nontrivial Lagrange anchor. The Lagrange anchor for the interacting equations can be sought for from the basic relation (3.10) by a perturbation theory in the interaction constant, but the existence is not guaranteed. Obstructions can appear, in principle, to the existence of the perturbative solution for the Lagrange anchor. If any obstruction appeared, it could have sense to identify that, because it would mean that the interacting higher spin fields cannot be quantized in the way consistent with the free limit. If no obstructions appeared, the Lagrange anchor would make possible the quantization of the interacting higher-spin fields, that remains an unsolved problem for many years.

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References

- [1] M. A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, Ann. Phys. (N.Y.) **190** (1989) 59-106.
- [2] M. A. Vasiliev, *Actions, Charges and Off-Shell Fields in the Unfolded Dynamics Approach*, Int. J. Geom. Meth. Mod. Phys. **3** (2006) 37-80.
- [3] M. A. Vasiliev, *Higher spin gauge theories in various dimensions*, Fortsch. Phys. **52** (2004) 702-717.
- [4] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, In Proc. of the First Solvay Workshop on Higher-Spin Gauge Theories, May 12-14, Brussels; 2004arXiv:0503128 [hep-th].
- [5] P. O. Kazinski, S. L. Lyakhovich and A. A. Sharapov, *Lagrange structure and quantization*, JHEP 0507:076 (39p.).
- [6] S. L. Lyakhovich and A. A. Sharapov, *Schwinger-Dyson equation for non-Lagrangian field theory*, JHEP 0602:007 (41p.).
- [7] S. L. Lyakhovich and A. A. Sharapov, *Quantizing non-Lagrangian gauge theories: An augmentation method*, JHEP 0701:047 (39p.).

- [8] D. S. Kaparulin, S. L. Lyakhovich and A.A. Sharapov, *Rigid Symmetries and Conservation Laws in Non-Lagrangian Field Theory*, J. Math. Phys. **51** (2010) 082902-1-082902-22.
- [9] S. L. Lyakhovich and A. A. Sharapov, *Quantization of Donaldson-Uhlenbeck-Yau theory*, Phys. Lett. **B656** (2007) 265-271.
- [10] G. Barnich and M. Grigoriev, *A Poincare lemma for sigma models of AKSZ type*, arXiv:0905.0547 [math-ph].
- [11] D. S. Kaparulin, *Quantization of field theories in unfolded representation*, Master Thesis, Tomsk, May 2009, in Russian, unpublished.
- [12] R. D'Auria and P. Fre, *Geometric supergravity in D=11 and its hidden supergroup*, Nucl. Phys. **B201** (1982) 101-140.
- [13] P. Fre and P. A. Grassi, *Free Differential Algebras, Rheonomy, and Pure Spinors*, arXiv: 0801.3076 [hep-th].
- [14] R. L. Bryant, S. S. Chern, R. B. Gardner, H. H. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, Springer-Verlag, New York, 1991.
- [15] O. V. Shaynkman and M. A. Vasiliev, *Scalar Field in Any Dimension From the Higher Spin Gauge Theory Perspective*, Theor. Math. Phys. **123** (2000) 683-700.